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# Blow-up phenomenon for a periodic rod equation

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**Abstract**
In this paper, firstly we find the optimal constant for a kind of Sobolev inequality on the unit circle via Fourier series. Then we apply the optimal constant on a nonlinear rod equation to give sufficient conditions on the initial datum, which guarantee finite-time singularity formation for the corresponding solution.

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## 1. Introduction

Although a rod is always three dimensional, if its diameter is much less than the axial length scale, one-dimensional equations can give a good description of the motion of the rod. Recently, Dai [20] derived a new (one-dimensional) nonlinear dispersive equation including extra nonlinear terms involving second-order and third-order derivatives for a compressible hyperelastic material. The equation reads

$$v_\tau + \sigma_1 v v_\xi + \sigma_2 v_{\xi\xi\tau} + \sigma_3(2v_\xi v_{\xi\xi} + v v_{\xi\xi\xi}) = 0,$$

where $v(\xi,\tau)$ represents the radial stretch relative to a pre-stressed state, $\sigma_1 \neq 0, \sigma_2 < 0$ and $\sigma_3 \leqslant 0$ are constants determined by the pre-stress and the material parameters. If one introduces the following transformations:

$$\tau = \frac{3\sqrt{-\sigma_2}}{\sigma_1}t, \qquad \xi = \sqrt{-\sigma_2}x,$$

then the above equation turns into

$$u_t - u_{txx} + 3uu_x = \gamma(2u_x u_{xx} + uu_{xxx}), \tag{1.1}$$

where $\gamma = 3\sigma_3/(\sigma_1\sigma_2)$. In [21], the authors derived that the value range of $\gamma$ is from $-29.4760$ to 3.4174 for some special compressible materials. From the mathematical view point, we regard $\gamma$ as a real number.

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When $\gamma = 1$ in (1.1), we recover the shallow water (Camassa–Holm) equation derived physically by Camassa and Holm in [5] by approximating directly the Hamiltonian for Euler's equations in the shallow water regime, where $u(x, t)$ represents the free surface above a flat bottom. Recently, the alternative derivations of the Camassa–Holm equation as a model for water waves, respectively, as the equation for geodesic flow on the diffeomorphism group of the circle were presented by Johnson [24] and, respectively, by Constantin and Kolev [12]. The geometric interpretation is important because it can be used to prove that the least action principle holds for the Camassa–Holm equation (cf [13]). It is worth to point out that a fundamental aspect of the Camassa–Holm equation, the fact that it is a completely integrable system, was shown in [7, 17] for the periodic case and [1, 8, 11] for the nonperiodic case. Some satisfactory results have been obtained for this shallow water equation recently. Local well-posedness for the initial datum $u_0(x) \in H^s$ with $s > 3/2$ was proved by several authors (see [26, 28, 31]). For the initial data with lower regularity, we refer to Molinet's paper [29] and also the recent paper [4]. Moreover, wave breaking for a large class of initial data has been established in [14, 15, 26, 27, 34]. However, in [32], global existence of weak solutions is proved but uniqueness is obtained only under an *a priori* assumption that is known to hold only for initial data $u_0(x) \in H^1$ such that $u_0 - u_{0xx}$ is a sign-definite Radon measure (under this condition, global existence and uniqueness was shown in [16] also). Also it is worth to note that global conservative solutions are constructed for any initial data in $H^1$ by Bressan and Constantin [4] recently. In [2, 18], it was proved that all solitary waves (peaked when $c_0 = 0$ or smooth when $c_0 \neq 0$) are solitons. The stabilities of the solitons are proved in [18, 19], respectively. Recently, in [23], among others, Himonas, Misiołek, Ponce and the second author showed the infinite propagation speed for the Camassa–Holm equation in the sense that a strong solution of the Cauchy problem with compact initial profile cannot be compactly supported at any later time unless it is the zero solution, which is an improvement of a first result in this direction obtained in [9].

If $\gamma = 0$, (1.1) is the BBM equation, a well-known model for surface waves in a canal [3], and its solutions are global.

For general $\gamma \in \mathbb{R}$, the rod equation (1.1) was studied sketchily by the Constantin and Strauss in [19] first. Local well-posedness of strong solutions to (1.1) was established by applying Kato's theory [25] and some sufficient conditions on the initial data were found to guarantee the finite blow-up of the corresponding solutions for spatially nonperiodic case. Later, in [36], the second author proved the well-posedness result in detail and various refined sufficient conditions on the initial data were found to guarantee the finite blow-up of the corresponding solutions for both spatially periodic and nonperiodic cases (see also some results in [33]). It should be mentioned that for $\gamma < 1$, (1.1) admits smooth solitary waves observed by Dai and Huo [21]. Let $u(x, t) = \phi(\xi)$, $\xi = x - ct$ be the solitary wave to (1.1). It was shown that $\phi(\xi)$ satisfies

$$\pm\xi = -\sqrt{-\gamma}\left(\frac{1}{2}\pi + \arcsin\frac{2\gamma\phi - (\gamma+1)c}{(1-\gamma)c}\right) - \ln\frac{(\sqrt{c(c-\phi)} + \sqrt{c(c-\gamma\phi)})^2}{(1-\gamma)c\phi}$$

for $\gamma < 0$ and

$$\pm\xi = \sqrt{\gamma}\ln\frac{(\sqrt{c-\gamma\phi)} - \sqrt{\gamma(c-\phi)})^2}{(1-\gamma)c} - \ln\frac{(\sqrt{c-\gamma\phi} + \sqrt{c-\phi})^2}{(1-\gamma)\phi}$$

for $0 < \gamma < 1$. In [19] (see [35] also), Constantin and Strauss proved the stability of these solitary waves by applying a general theorem established by Grillakis, Shatah and Strauss [22].

We now finish this introduction by outlining the rest of the paper. In section 2, we recall the local well-posedness for (1.1) with initial datum $u_0 \in H^s$, $s > 3/2$, and the lifespan of the corresponding solution is finite if and only if its first-order derivative blows up while the solution remains bounded, that is, the rod breaks. In section 3, we find the optimal constant for a kind of Sobolev inequality via Fourier series. Then we show blow-up of solutions to the nonlinear dispersive rod equation by applying this optimal constant.

## 2. Preliminaries

In this paper, we concentrate on the periodic case and $\mathbb{S} = \mathbb{R}/\mathbb{Z}$ denotes the unit circle. In [19, 36], it is proved that

**Theorem 2.1** [19, 36]. *Let the initial datum $u_0(x) \in H^s(\mathbb{S}), s > 3/2$. Then there exists $T = T(\|u_0\|_{H^s}) > 0$ and a unique solution u, which depends continuously on the initial datum $u_0$, to (1.1) such that*

$$u \in C([0, T); H^s(\mathbb{S})) \cap C^1([0, T); H^{s-1}(\mathbb{S})).$$

*Moreover, the following two quantities E and F are invariants with respect to time t for (1.1):*

$$\begin{cases} E(u)(t) = \int_{\mathbb{S}} \left(u^2(x,t) + u_x^2(x,t)\right) \mathrm{d}x \\ F(u)(t) = \int_{\mathbb{S}} \left(u^3(x,t) + \gamma u(x,t) u_x^2(x,t)\right) \mathrm{d}x. \end{cases}$$

Actually, the local well-posedness was proved for both periodic and nonperiodic case in the above paper.

The maximum value of $T$ in theorem 2.1 is called the lifespan of the solution, in general. If $T < \infty$, that is $\limsup_{t\uparrow T} \|u(\cdot,t)\|_{H^s} = \infty$, we say that the solution blows up in finite time. The following theorem tells us that the solution blows up if and only if the first-order derivative blows up.

**Theorem 2.2** [19, 36]. *Let $u_0(x) \in H^s(\mathbb{S}), s > 3/2$, and u be the corresponding solution to problem (1.1) with lifespan T. Then*

$$\sup_{x\in\mathbb{S}, 0\leqslant t<T} |u(x,t)| \leqslant C(\|u_0\|_{H^1}). \tag{2.1}$$

*T is bounded if and only if*

$$\liminf_{t\uparrow T} \inf_{x\in\mathbb{S}} \{\gamma u_x(x,t)\} = -\infty. \tag{2.2}$$

For $\gamma \neq 0$, we set

$$m(t) := \inf_{x\in\mathbb{S}} \left(u_x(x,t)\, \mathrm{sign}\{\gamma\}\right), \qquad t \geqslant 0, \tag{2.3}$$

where sign$\{a\}$ is the sign function of $a \in \mathbb{R}$ and we set $m_0 := m(0)$. Then for every $t \in [0, T)$ there exists at least one point $\xi(t) \in \mathbb{S}$ with $m(t) = u_x(\xi(t), t)$.

**Lemma 2.3** [19]. *Let $u(t)$ be the solution to (1.1) on $[0, T)$ with initial data $u_0 \in H^s(\mathbb{S})$, $s > 3/2$, as given by theorem 2.1. Then the function $m(t)$ is almost everywhere differentiable on $[0, T)$, with*

$$\frac{\mathrm{d}m(t)}{\mathrm{d}t} = u_{tx}(\xi(t), t), \qquad \text{a.e. on} \quad (0, T).$$

Consideration of the quantity $m(t)$ for wave breaking comes from an idea of Seliger [30] originally. The rigorous regularity proof is given in [15] for Camassa–Holm equation.

Set $Q^s = \left(1-\partial_x^2\right)^{s/2}$, then the operator $Q^{-2}$ can be expressed by

$$Q^{-2}f = G * f = \int_{\mathbb{T}} G(x-y)f(y)\,\mathrm{d}y$$

for any $f \in L^2(\mathbb{S})$ with

$$G(x) = \frac{\cosh(x-[x]-1/2)}{2\sinh(1/2)}, \tag{2.4}$$

where $[x]$ denotes the integer part of $x$. Then equation (1.1) can be rewritten as

$$u_t + \gamma u u_x + \partial_x Q^{-2}\left(\frac{3-\gamma}{2}u^2 + \frac{\gamma}{2}u_x^2\right) = 0. \tag{2.5}$$

Just as in [19, 36], it is easy to derive a equation for $m(t)$ from (2.5) as

$$\frac{\mathrm{d}m}{\mathrm{d}t} = -\frac{\gamma}{2}m^2 + \frac{3-\gamma}{2}u^2(\xi(t),t) - \left[G * \left(\frac{3-\gamma}{2}u^2 + \frac{\gamma}{2}u_x^2\right)\right](\xi(t),t) \tag{2.6}$$

a.e. on $(0,T)$, where $m(t)$ and $\xi(t)$ were defined in (2.3) and lemma 2.3.

If $\gamma = 3$, it turns out that (2.6) is a Riccati-type equation with negative initial data for any nonconstant $u_0$. So the solutions to (1.1) in periodic case definitely blow up in finite time with arbitrary nonconstant initial data $u_0$.

In what follows, we assume that $0 < \gamma < 3$.

## 3. The optimal constant for a kind of Sobolev inequality

Due to the following inequality

$$0 \leqslant \left(\int_{\mathbb{S}} f(x)\,\mathrm{d}x\right)^2 \leqslant \|f\|_{H^1(\mathbb{S})}^2 \qquad \text{for any} \qquad f \in H^1(\mathbb{S}),$$

we know the norm

$$\|f\|_{H^1(\mathbb{S})}^2 + d\bar{f}^2 \qquad \text{for} \qquad d > -1$$

is equivalent to the $H^1$-norm's square of $f$, where $\bar{f} = \int_{\mathbb{S}} f(x)\,\mathrm{d}x$.

It is easy to find that the integral $\int_{\mathbb{S}} u(x,t)$ is an invariant with respect to time $t$, if $u(x,t)$ is a solution to (1.1). The purpose of this section is to establish the best constant for a kind of Sobolev inequality. The main theorem reads

**Theorem 3.1.** *For any $f \in H^1(\mathbb{S})$, we have*

$$\|f\|_{L^\infty(\mathbb{S})}^2 \leqslant \left(\frac{1}{e-1} + \frac{1-d}{2(1+d)}\right)\left(\|f\|_{H^1(\mathbb{S})}^2 + d\bar{f}^2\right), \tag{3.1}$$

*for any $d > -1$. Moreover, the constant is optimal and can be achieved by function $f = G(x-x_0) - \frac{d}{1+d}$ for some $x_0$, where $G(x)$ is the Green's function to $\left(1-\partial_x^2\right)$ and defined by (2.4).*

**Proof.** Since $f$ is a periodic function and the above inequality is rotation invariant, it is sufficient to consider it in one period $[0,1]$ and expand $f$ as a cosine series

$$f(x) = \sum_{n=0}^{\infty} a_n \cos(2\pi n x),$$

for $x \in [0,1]$.

First by direct computation,

$$\int_{\mathbb{S}} f^2(x)\,\mathrm{d}x = \int_{\mathbb{S}} \left(\sum_{n=0}^{\infty} a_n \cos(2\pi n x)\right)^2 \mathrm{d}x = a_0^2 + \sum_{n=1}^{\infty} \frac{a_n^2}{2} \tag{3.2}$$

and

$$\int_{\mathbb{S}} f_x^2(x)\,\mathrm{d}x = \int_{\mathbb{S}} \left(\sum_{n=1}^{\infty} 2n\pi a_n \sin(2\pi n x)\right)^2 \mathrm{d}x = \sum_{n=1}^{\infty} 2n^2\pi^2 a_n^2. \tag{3.3}$$

Now, let

$$A = \sum_{n=1}^{\infty} \frac{2}{1+4\pi^2 n^2}.$$

Thanks to Cauchy–Schwartz inequality, we have

$$\begin{aligned}
\|f\|_{L^\infty(\mathbb{S})}^2 &\leqslant \left(\sum_{n=0}^{\infty} |a_n|\right)^2 \\
&\leqslant \left(1 + \frac{1}{(1+d)A}\right)\left(\sum_{n=1}^{\infty} |a_n|\right)^2 + (1+(1+d)A)|a_0|^2 \\
&\leqslant \left(A + \frac{1}{1+d}\right)\sum_{n=1}^{\infty} \frac{1+4\pi^2 n^2}{2}|a_n|^2 + 1 + (1+d)A)|a_0|^2 \\
&= \left(A + \frac{1}{1+d}\right)\left(\left(|a_0|^2 + \sum_{n=1}^{\infty} \frac{1+4\pi^2 n^2}{2}|a_n|^2\right) + d|a_0|^2\right) \\
&= \left(A + \frac{1}{1+d}\right)\left(\|f\|_{H^1(\mathbb{S})}^2 + d\bar{f}^2\right),
\end{aligned}$$

where we used (3.2) and (3.3). The equality can be achieved, if we choose

$$a_n = \frac{2}{1+4\pi^2 n^2}, \qquad a_0 = \frac{1}{1+(1+d)A}\sum_{n=1}^{\infty} |a_n| = \frac{A}{1+(1+d)A}.$$

From the following identity:

$$\frac{1}{x} + \sum_{n=1}^{\infty} \frac{2x}{x^2+4\pi^2 n^2} = \frac{\cosh(x/2)}{2\sinh(x/2)}, \qquad \forall\, x \neq 0, \tag{3.4}$$

by taking $x = 1$, we get

$$A + \frac{1}{1+d} = \frac{1}{e-1} + \frac{1-d}{2(1+d)}.$$

Using (3.4) again, we get the achieved function. This finishes the proof. □

**Remark 3.1.** The proof also can be done for the full Fourier series, i.e., $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(2\pi n x) + b_n \sin(2\pi n x)$.

**Remark 3.2.** By taking $d = 0$ in theorem 2.1, we recover the optimal constant for the following inequality:

$$\|f\|_{L^\infty(\mathbb{S})}^2 \leqslant \frac{\cosh(1/2)}{2\sinh(1/2)}\|f\|_{H^1(\mathbb{S})}^2,$$

which is proved in [34] via a variational method. It is worth mentioning that the method here is much simpler than that in [34].

When $d = 1$, the inequality (3.1) reduces to

$$\|f\|^2_{L^\infty(\mathbb{S})} \leqslant \frac{1}{e-1}\left(\|f\|^2_{H^1(\mathbb{S})} + \bar{f}^2\right),$$

which is an improvement for the previous one (proved in [34])

$$\|f\|^2_{L^\infty(\mathbb{S})} \leqslant \|f\|^2_{H^1(\mathbb{S})} + \bar{f}^2.$$

## 4. Blow-up criteria

After local well-posedness of strong solutions (see theorem 2.1) is established, the next question is whether this local solution can exist globally. As far as we know, the only available global existence results are for the case $\gamma = 1$: see the paper by Constantin [6] for a PDE approach, and the paper by Constantin and McKean [17] for an approach based on the integrable structure of the equation. If the solution exists only for finite time, how about the behavior of the solution when it blows up? What induces the blow-up? On the other hand, to find sufficient conditions to guarantee the finite time blow-up or global existence is of great interest, especially for sufficient conditions added on the initial data.

Before we write the main theorem of this section, let us recall the following inequality proved in [37]: for all $f \in H^1(\mathbb{S})$

$$G * \left(f^2 + \tfrac{1}{2}f_x^2\right)(x) \geqslant Cf^2(x), \tag{4.1}$$

with

$$C = \frac{1}{2} + \frac{\arctan(\sinh(1/2))}{2\sinh(1/2) + 2\arctan(\sinh(1/2))\sinh^2(1/2)}.$$

For simplicity, for $d > -1$, we take the following notation:

$$C_d := \frac{1}{e-1} + \frac{1-d}{2(1+d)} \qquad \text{and} \qquad \|f\|_{H^1_d} := \|f\|^2_{H^1(\mathbb{S})} + d\bar{f}^2.$$

The main theorem of this section is as follows.

**Theorem 4.1.** *Let* $0 < \gamma < 3$, $d > -1$. *Assume that* $u_0 \in H^2(\mathbb{S})$ *satisfies* $m_0 < 0$ *and*

$$\begin{cases} m_0^2 > \dfrac{(6-\gamma-\sqrt{12\gamma-3\gamma^2})C_d}{2\gamma}\|u_0\|^2_{H^1_d(\mathbb{S})}, & \text{if} \quad 0 < \gamma \leqslant \gamma_0, \\ m_0^2 > \dfrac{(3-(1+2C)\gamma)C_d}{\gamma}\|u_0\|^2_{H^1_d(\mathbb{S})}, & \text{if} \quad \gamma_0 < \gamma \leqslant 1, \\ m_0^2 > \dfrac{(3-\gamma)(1-C)C_d}{\gamma}\|u_0\|^2_{H^1_d(\mathbb{S})}, & \text{if} \quad 1 < \gamma \leqslant \gamma_1, \\ m_0^2 > \dfrac{(6-\gamma-\sqrt{12\gamma-3\gamma^2})C_d}{2\gamma}\|u_0\|^2_{H^1_d(\mathbb{S})}, & \text{if} \quad \gamma_1 < \gamma < 3, \end{cases}$$

*where*

$$\gamma_0 = \frac{3}{4C^2 + 2C + 1}, \qquad \gamma_1 = \frac{3C^2}{C^2 - C + 1},$$

*while* $C$ *is the optimal constant in (4.1). Then the life span* $T > 0$ *of the corresponding solution to (1.1) is finite.*

**Remark 4.1.** If $\gamma = 1$, equation (1.1) is reduced to the Camassa–Holm equation, while the condition is

$$m_0 < 0, \qquad m_0^2 > 2(1-C)C_d\|u_0\|_{H_d^1},$$

which is an improvement of $m_0^2 > \left(\|u_0\|_{H^1}^2 + \bar{u_0}^2\right)$ proved in [34].

**Remark 4.2.** Where $\gamma_0 \approx 0.521$ and $\gamma_1 \approx 2.555$ are solutions, respectively, to

$$-\frac{1}{2} + \frac{1}{2}\sqrt{\frac{12-3\gamma}{\gamma}} = 2C \qquad \text{and} \qquad \left(-\frac{1}{2} + \frac{1}{2}\sqrt{\frac{12-3\gamma}{\gamma}}\right)\gamma = (3-\gamma)C.$$

The conditions for $0 < \gamma \leqslant \gamma_0$ and $\gamma_1 < \gamma < 3$ are established in [36] first. The cases for $\gamma < 0$ and $\gamma > 3$ were also discussed in [36].

First, we have the following blow-up result for a Riccati-type ordinary differential equation.

**Lemma 4.2.** *Assume that a differentiable function $y(t)$ satisfies*

$$y'(t) \leqslant -Cy^2(t) + K, \tag{4.2}$$

*with constants $C, K > 0$. If the initial datum $y(0) = y_0 < -\sqrt{\frac{K}{C}}$, then the solution to (4.2) goes to $-\infty$ in finite time.*

It is easy to prove. For the details, please refer to [37].

Now, let us start the proof for the main theorem from (2.6). We will treat it case by case.

(i) $0 < \gamma \leqslant \gamma_0$.

By the representation of $G$, we have

$$\begin{aligned}\left[G * \left(\frac{3-\gamma}{2}u^2 + \frac{\gamma}{2}u_x^2\right)\right](x,t) &= \frac{1}{2\sinh\left(\frac{1}{2}\right)}\int_0^x \frac{e^{x-\eta-\frac{1}{2}} + e^{\frac{1}{2}+\eta-x}}{2}\left(\frac{3-\gamma}{2}u^2(\eta,t) + \frac{\gamma}{2}u_x^2(\eta,t)\right)d\eta \\ &\quad + \frac{1}{2\sinh\left(\frac{1}{2}\right)}\int_x^1 \frac{e^{x-\eta+\frac{1}{2}} + e^{\eta-x-\frac{1}{2}}}{2}\left(\frac{3-\gamma}{2}u^2(\eta,t) + \frac{\gamma}{2}u_x^2(\eta,t)\right)d\eta.\end{aligned} \tag{4.3}$$

Direct computation yields that

$$\begin{aligned}\int_0^x e^{-\eta}\left(\frac{\gamma}{2}\alpha^2u^2(\eta,t) + \frac{\gamma}{2}u_x^2(\eta,t)\right)d\eta &\geqslant -\int_0^x e^{-\eta}\gamma\alpha u(\eta,t)u_x(\eta,t)\,d\eta \\ &= -e^{-\eta}\frac{\gamma\alpha}{2}u^2(\eta,t)\Big|_0^x - \int_0^x e^{-\eta}\frac{\gamma\alpha}{2}u^2(\eta,t)\,d\eta\end{aligned}$$

holds for any $\alpha > 0$. We have

$$\int_0^x e^{-\eta}\left((\alpha^2+\alpha)\frac{\gamma}{2}u^2(\eta,t) + \frac{\gamma}{2}u_x^2(\eta,t)\right)d\eta \geqslant -\frac{\alpha\gamma}{2}\,e^{-\eta}u^2(\eta,t)\Big|_0^x.$$

Now we let

$$\alpha^2 + \alpha = \frac{3-\gamma}{\gamma},$$

which has one positive root $\alpha_0$ with

$$\alpha_0 = -\frac{1}{2} + \frac{1}{2}\sqrt{\frac{12-3\gamma}{\gamma}}. \tag{4.4}$$

Therefore

$$\int_0^x \mathrm{e}^{-\eta}\left(\frac{3-\gamma}{2}u^2(\eta,t) + \frac{\gamma}{2}u_x^2(\eta,t)\right)\mathrm{d}\eta \geqslant -\frac{\alpha_0\gamma}{2}\,\mathrm{e}^{-\eta}u^2(\eta,t)\Big|_0^x.$$

Moreover, from (4.3), just use the above trick for each term, then one obtains that

$$\left[G * \left(\frac{3-\gamma}{2}u^2 + \frac{\gamma}{2}u_x^2\right)\right](\xi(t),t) \geqslant \frac{\alpha_0\gamma}{2}u^2(\xi(t),t). \tag{4.5}$$

The approach used above parallels that presented in the paper by Constantin [10] for the Camassa–Holm equation ($\gamma = 1$).

Now combining (2.6) and (4.5) together, we have

$$\begin{aligned}\frac{\mathrm{d}m}{\mathrm{d}t} &\leqslant -\frac{\gamma}{2}m^2 + \frac{6-\gamma-\sqrt{12\gamma-3\gamma^2}}{4}u^2(\xi(t),t)\\ &\leqslant -\frac{\gamma}{2}m^2 + \frac{(6-\gamma-\sqrt{12\gamma-3\gamma^2})C_d}{4}\|u_0\|_{H^1}^2,\end{aligned} \tag{4.6}$$

where we used (3.1) and the conservation of $H_d^1$-norm.

If

$$m_0 < -\left(\frac{(6-\gamma-\sqrt{12\gamma-3\gamma^2})C_d}{2\gamma}\right)^{1/2}\|u_0\|_{H_d^1},$$

then the solution $m(t)$ to (4.6) goes to $-\infty$ in finite time by applying lemma 4.2.

(ii) $\gamma_0 < \gamma \leqslant 1$

By direct computation, we have

$$\begin{aligned}G * \left(\frac{3-\gamma}{2}u^2 + \frac{\gamma}{2}u_x^2\right)(x,t) &= \gamma G * \left(u^2 + \frac{1}{2}u_x^2\right)(x,t) + \frac{3(1-\gamma)}{2}G * u^2(x,t)\\ &\geqslant \gamma G * \left(u^2 + \frac{1}{2}u_x^2\right)(x,t)\\ &\geqslant \gamma C u^2(x,t),\end{aligned} \tag{4.7}$$

where $C$ is the constant in (4.1).

Putting (4.7) into (2.6), one has

$$\begin{aligned}\frac{\mathrm{d}m}{\mathrm{d}t} &\leqslant -\frac{\gamma}{2}m^2 + \left(\frac{3-\gamma}{2} - \gamma C\right)u^2(\xi(t),t)\\ &\leqslant -\frac{\gamma}{2}m^2 + \frac{(3-(1+2C)\gamma)C_d}{2}\|u_0\|_{H^1}^2.\end{aligned}$$

Due to lemma 4.2, it is easy to see that the condition given in theorem 4.1 guarantees the blow-up of solutions.

(iii) $1 < \gamma \leqslant \gamma_1$

Similar to the case (ii),

$$\begin{aligned}G * \left(\frac{3-\gamma}{2}u^2 + \frac{\gamma}{2}u_x^2\right)(x,t) &= \frac{3-\gamma}{2}G * \left(u^2 + \frac{1}{2}u_x^2\right)(x,t) + \frac{3(\gamma-1)}{4}G * u_x^2(x,t)\\ &\geqslant \frac{3-\gamma}{2}G * \left(u^2 + \frac{1}{2}u_x^2\right)(x,t)\\ &\geqslant \frac{3-\gamma}{2}C_0u^2(x,t),\end{aligned}$$

which reduces (2.6) to

$$\frac{\mathrm{d}m}{\mathrm{d}t} \leqslant -\frac{\gamma}{2}m^2 + \frac{(3-\gamma)(1-C_0)\cosh(1/2)}{4\gamma\sinh(1/2)}\|u_0\|_{H^1}^2.$$

we can get the blow-up result in this case.

(iv) $\gamma_1 < \gamma < 3$

A similar argument for the first case $0 < \gamma < \gamma_0$ can be used here.

The proof of theorem 4.1 is complete.

**Remark 4.3.** From the proof, the reader may ask why we do not find the optimal constant $C$ for the following convolution problem directly:

$$G * \left(\frac{3-\gamma}{2}u^2 + \frac{\gamma}{2}u_x^2\right)(x) \geqslant Cu^2(x).$$

The answer is that we do not have any effective method to solve this problem at present. We hope we can deal with it in the near future.

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